Finding Plans and Heuristics with Spectral Graph Theory

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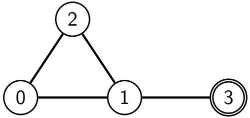
Undirected Simple Graphs + Goal $G = \langle V, E, g \rangle$



Matrices for the Graph

$$\text{Adjacency } A = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} & 1 & 1 & \\ 1 & & 1 & 1 \\ 1 & 1 & & \\ & 1 & & \end{bmatrix} \end{array}$$

$$\text{Degree } D = \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 2 & & & \\ & 3 & & \\ & & 2 & \\ & & & 1 \end{bmatrix} \end{array}$$

$$\text{Laplacian } L = D - A = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 2 & -1 & -1 & \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 2 & \\ & -1 & & 1 \end{bmatrix} \end{array}$$

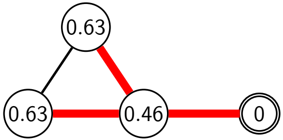
$$\text{Dirichlet } L_g = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 2 & -1 & -1 & \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 2 & \\ & -1 & & 1 \end{bmatrix} \end{array}$$

Spectral Algorithm for Finding Plans [Steinerberger, 2021]

1. Compute eigenvector $\mathbf{v} \in \mathbb{R}^{|V|-1}$ such that

$$L_g \mathbf{v} = \lambda_0 \mathbf{v}$$

2. Use v_i as heuristic for vertex i
3. Follow greedily



This will reach the goal!

their proof "*follows classical arguments from the continuous setting*"

It is a Solution to the Occupation Measure LP

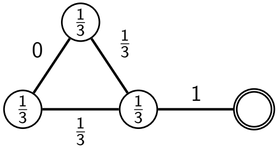
Looks like Operator Counting Net-change Heuristic LP

$$\min_{\mathbf{x}} \sum_{\langle i,j\rangle \in E} x_{i,j} \quad \text{s.t.}$$

$$out(i) - in(i) = \alpha(i) \quad \forall i \in V \setminus \{g\}$$

$$in(g) = 1$$

$$x_{i,j} \geq 0 \quad \forall \langle i,j \rangle \in E$$



Feasible solution induces proper policy (reaches goal with probability 1) [Puterman, 2005]

For us ☞ feasible solution induces plans

Theorem

$\mathbf{v}$ *is a feasible solution for the OM LP.*

Proof.

Sketch. $L_g\mathbf{v} \in \mathbb{R}^{|V \setminus \{g\}|}$ such that

$$(L_g\mathbf{v})_i = \underbrace{\sum_{\substack{\langle i,j\rangle \in E \\ i \prec j}} \underbrace{v_i - v_j}_{x_{i,j}}}_{out(i)} - \underbrace{\sum_{\substack{\langle i,j\rangle \in E \\ j \prec i}} \underbrace{v_j - v_i}_{x_{j,i}}}_{in(i)} = \underbrace{\lambda_0 v_i}_{\alpha(i)}$$

□

Corollary

By following $\mathbf{v}$ *greedily you follow a proper policy and are guaranteed to reach the goal.*

Let $h(i) = \kappa v_i$ s.t. $\kappa\left(v_i - v_j\right) \leq 1 \quad \forall\langle i,j\rangle \in E$ Note $\kappa \geq 1$

Theorem

h satisfies

$$h(i) \leq 1 + h(j) \quad \forall\langle i,j\rangle \in E$$
$$h(g) = 0$$

Proof.

Sketch.

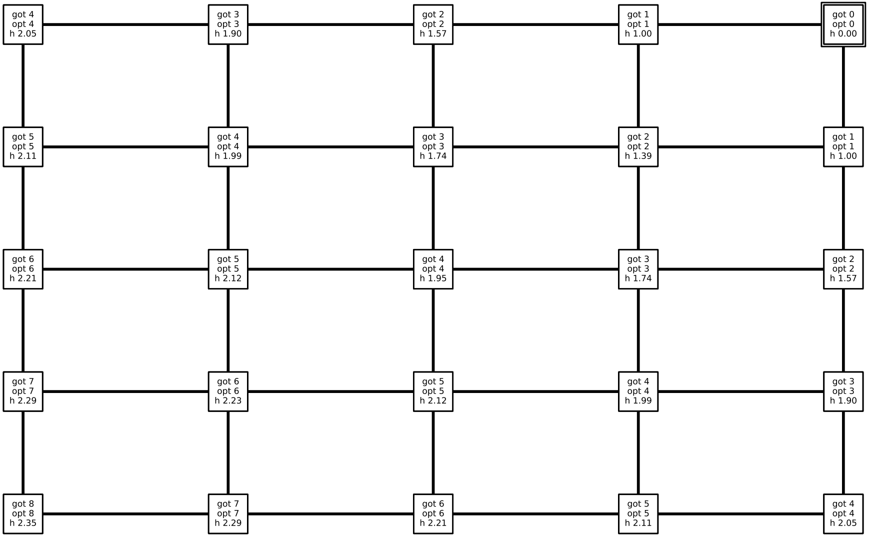
$$h(i) = \kappa v_i = \kappa(v_i - v_j) + \kappa v_j \leq 1 + h(j)$$

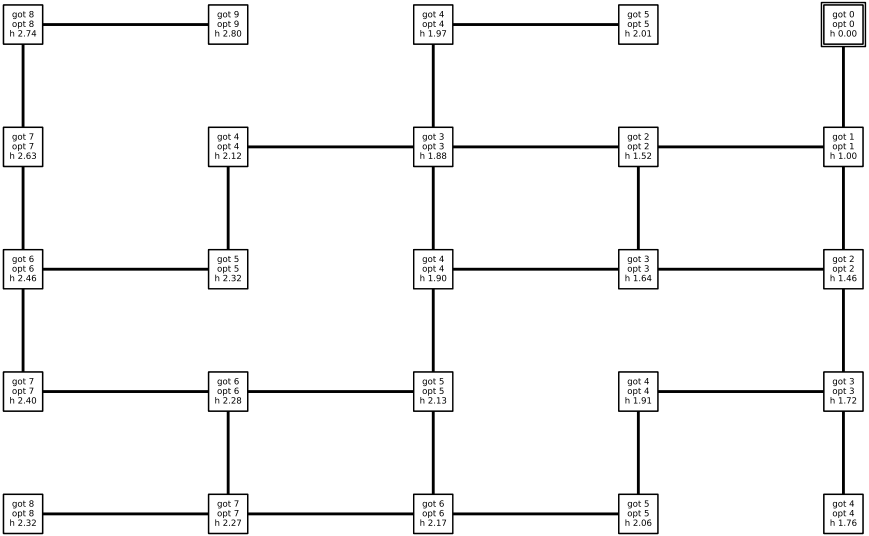
□

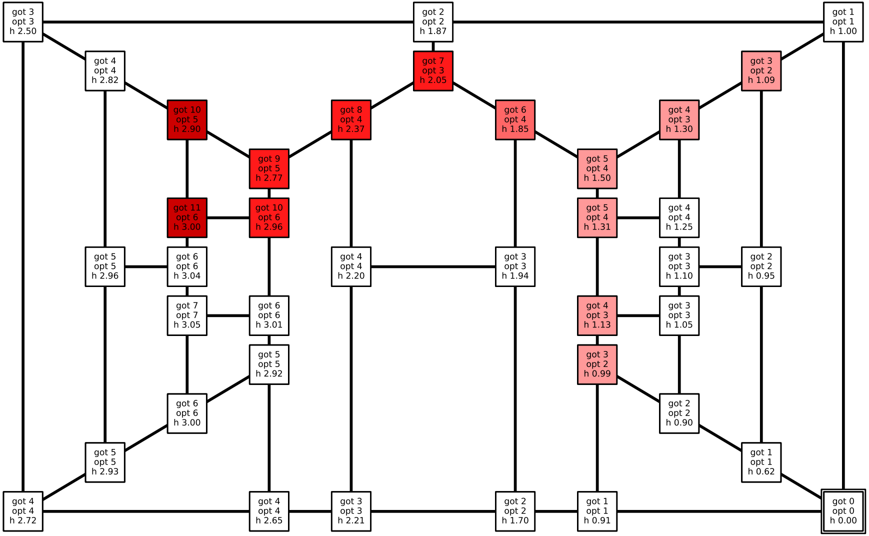
Corollary

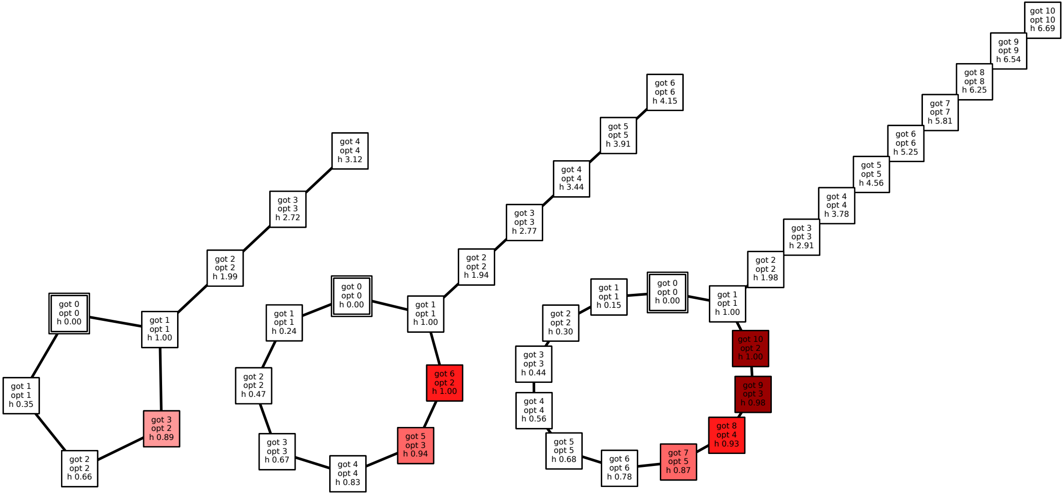
h is a consistent, goal-aware heuristic. [Pommerening et al., 2015]

Note: relation to OM LP's dual!


got 4
opt 4
h 2.05
got 3
opt 3
h 1.90
got 2
opt 2
h 1.57
got 1
opt 1
h 1.00
got 0
opt 0
h 0.00
got 5
opt 5
h 2.11
got 4
opt 4
h 1.99
got 3
opt 3
h 1.74
got 2
opt 2
h 1.39
got 1
opt 1
h 1.00
got 6
opt 6
h 2.21
got 5
opt 5
h 2.12
got 4
opt 4
h 1.95
got 3
opt 3
h 1.74
got 2
opt 2
h 1.57
got 7
opt 7
h 2.29
got 6
opt 6
h 2.23
got 5
opt 5
h 2.12
got 4
opt 4
h 1.99
got 3
opt 3
h 1.90
got 8
opt 8
h 2.35
got 7
opt 7
h 2.29
got 6
opt 6
h 2.21
got 5
opt 5
h 2.11
got 4
opt 4
h 2.05

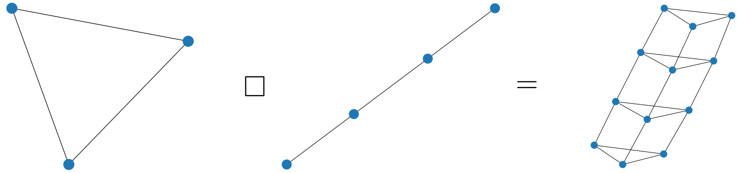

got 8 opt 8 h 2.74
got 9 opt 9 h 2.80
got 4 opt 4 h 1.97
got 5 opt 5 h 2.01
got 0 opt 0 h 0.00
got 7 opt 7 h 2.63
got 4 opt 4 h 2.12
got 3 opt 3 h 1.88
got 2 opt 2 h 1.52
got 1 opt 1 h 1.00
got 6 opt 6 h 2.46
got 5 opt 5 h 2.32
got 4 opt 4 h 1.90
got 3 opt 3 h 1.64
got 2 opt 2 h 1.46
got 7 opt 7 h 2.40
got 6 opt 6 h 2.28
got 5 opt 5 h 2.13
got 4 opt 4 h 1.91
got 3 opt 3 h 1.72
got 8 opt 8 h 2.32
got 7 opt 7 h 2.27
got 6 opt 6 h 2.17
got 5 opt 5 h 2.06
got 4 opt 4 h 1.76


got 3 opt 3 h 2.50
got 2 opt 2 h 1.87
got 1 opt 1 h 1.00
got 4 opt 4 h 2.82
got 7 opt 3 h 2.05
got 3 opt 2 h 1.09
got 10 opt 5 h 2.90
got 8 opt 4 h 2.37
got 6 opt 4 h 1.85
got 4 opt 3 h 1.30
got 9 opt 5 h 2.77
got 5 opt 4 h 1.50
got 11 opt 6 h 3.00
got 10 opt 6 h 2.96
got 5 opt 4 h 1.31
got 4 opt 4 h 1.25
got 5 opt 5 h 2.96
got 6 opt 6 h 3.04
got 4 opt 4 h 2.20
got 3 opt 3 h 1.94
got 3 opt 3 h 1.10
got 2 opt 2 h 0.95
got 7 opt 7 h 3.05
got 6 opt 6 h 3.01
got 4 opt 3 h 1.13
got 3 opt 3 h 1.05
got 5 opt 5 h 2.92
got 3 opt 2 h 0.99
got 6 opt 6 h 3.00
got 2 opt 2 h 0.90
got 5 opt 5 h 2.93
got 1 opt 1 h 0.62
got 4 opt 4 h 2.72
got 4 opt 4 h 2.65
got 3 opt 3 h 2.21
got 2 opt 2 h 1.70
got 1 opt 1 h 0.91
got 0 opt 0 h 0.00


got 0 opt 0 h 0.00
got 1 opt 1 h 1.00
got 2 opt 2 h 1.99
got 3 opt 3 h 2.72
got 4 opt 4 h 3.12
got 1 opt 1 h 0.35
got 2 opt 2 h 0.66
got 3 opt 2 h 0.89
got 0 opt 0 h 0.00
got 1 opt 1 h 0.24
got 2 opt 2 h 0.47
got 3 opt 3 h 0.67
got 4 opt 4 h 0.83
got 5 opt 3 h 0.94
got 6 opt 2 h 1.00
got 1 opt 1 h 1.00
got 2 opt 2 h 1.94
got 3 opt 3 h 2.77
got 4 opt 4 h 3.44
got 5 opt 5 h 3.91
got 6 opt 6 h 4.15
got 0 opt 0 h 0.00
got 1 opt 1 h 0.15
got 2 opt 2 h 0.30
got 3 opt 3 h 0.44
got 4 opt 4 h 0.56
got 5 opt 5 h 0.68
got 6 opt 6 h 0.78
got 7 opt 5 h 0.87
got 8 opt 4 h 0.93
got 9 opt 3 h 0.98
got 10 opt 2 h 1.00
got 1 opt 1 h 1.00
got 2 opt 2 h 1.98
got 3 opt 3 h 2.91
got 4 opt 4 h 3.78
got 5 opt 5 h 4.56
got 6 opt 6 h 5.25
got 7 opt 7 h 5.81
got 8 opt 8 h 6.25
got 9 opt 9 h 6.54
got 10 opt 10 h 6.69

Getting transition system's matrix and finding an eigenvector is not realistic. . .

Graph Operations that Preserve Eigenvectors



if G has eigenvectors $\mathbf{x}_1, \ldots \mathbf{x}_m$ and H has eigenvectors $\mathbf{y}_1, \ldots \mathbf{y}_m$ then

$$G \square H \text{ has eigenvectors } \mathbf{x}_i \otimes \mathbf{y}_j \text{ for each } i, j$$

But □ does not give us much power. . .

Summary

1. We can find plans with the spectral method
2. This works because it finds a feasible solution to the Occupation Measure LP
3. The eigenvector also induces a consistent, goal-aware heuristic
4. Steinerberger [2021] suggests sticking with Dijkstra for now...

What's Next?

Would be nice to find a way to combine graphs that approximately preserves the required eigenvector...

Pommerening, F.; Helmert, M.; Röger, G.; and Seipp, J. 2015. From Non-Negative to General Operator Cost Partitioning. *Proc. of AAAI Conf. on Artificial Intelligence*, 3335–3341.

Puterman, M. L. 2005. *Markov Decision Processes*. Wiley-Interscience.

Steinerberger, S. 2021. A spectral approach to the shortest path problem. *Linear Algebra and its Applications*, 620: 182–200.