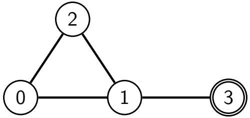
Finding Plans and Heuristics with Spectral Graph Theory

Johannes Schmalz

Universität des Saarlandes

11th November
HSDIP 2025

Connected Undirected Simple Graphs + Goal $G = \langle V, E, g \rangle$



Matrices for the Graph

$$\text{Adjacency } A = \begin{array}{c c} & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} & 1 & 1 & \\ 1 & & 1 & 1 \\ 1 & 1 & & \\ & 1 & & \end{bmatrix} \end{array} \qquad \text{Degree } D = \begin{array}{c c} & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 2 & & & \\ & 3 & & \\ & & 2 & \\ & & & 1 \end{bmatrix} \end{array}$$

$$\text{Laplacian } L = D - A = \begin{array}{c c} & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 2 & -1 & -1 & \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 2 & \\ & -1 & & 1 \end{bmatrix} \end{array} \qquad \text{Dirichlet } L_g = \begin{array}{c c} & \begin{array}{cccc} 0 & 1 & 2 & \cancel{3} \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ \cancel{3} \end{array} & \begin{bmatrix} 2 & -1 & -1 & \\ -1 & 3 & -1 & \cancel{-1} \\ -1 & -1 & 2 & \\ & \cancel{-1} & & \cancel{1} \end{bmatrix} \end{array}$$

Spectral Algorithm for Finding Plans [Steinerberger, 2021]

1. Compute eigenvector $\mathbf{v} \in \mathbb{R}^{|V|-1}$ such that

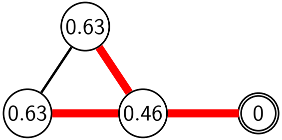
$$L_g \mathbf{v} = \lambda_0 \mathbf{v}$$

2. Treat v_i as heuristic for vertex i

   ☞ this heuristic is descending
   $\forall i \in V \setminus \{g\}\ \exists \langle i, j \rangle \in E$ s.t. $v_j < v_i$

3. Follow greedily

   ☞ guaranteed to reach the goal [Seipp et al., 2016]



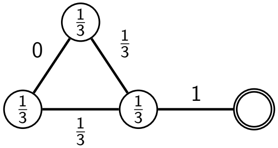
Steinerberger's proof "*follows classical arguments from the continuous setting*"

Eigenvector describes feasible flow

It is a Solution to the Occupation Measure LP

Looks like Operator Counting Net-change Heuristic LP

$$\min_{\mathbf{x}} \sum_{\langle i,j\rangle \in E} x_{i,j} \quad \text{s.t.}$$
$$out(i) - in(i) = \alpha(i) \quad \forall i \in V \setminus \{g\}$$
$$in(g) = 1$$
$$x_{i,j} \geq 0 \quad \forall \langle i,j\rangle \in E$$



Feasible solution induces proper policy (reaches goal with probability 1) [Puterman, 2005]

For us ☞ feasible solution induces plans

Theorem

$\mathbf{v}$ *is a feasible solution for the OM LP.*

Proof.

Sketch. $L_g\mathbf{v} \in \mathbb{R}^{|V\setminus\{g\}|}$ such that

$$(L_g\mathbf{v})_i = \underbrace{\sum_{\substack{\langle i,j\rangle\in E\\ v_i>v_j}} \underbrace{v_i - v_j}_{x_{i,j}}}_{out(i)} - \underbrace{\sum_{\substack{\langle j,i\rangle\in E\\ v_j>v_i}} \underbrace{v_j - v_i}_{x_{j,i}}}_{in(i)} = \underbrace{\lambda_0 v_i}_{\alpha(i)}$$

□

Corollary

By following $\mathbf{v}$ *greedily you follow a proper policy and are guaranteed to reach the goal.*

Eigenvector is a consistent, goal-aware heuristic

Heuristic $h(i) = \kappa v_i$

s.t. $\kappa\left(v_i - v_j\right) \leq 1 \quad \forall \langle i, j \rangle \in E$ Note $\kappa \geq 1$

Theorem

h satisfies

$$h(i) \leq 1 + h(j) \quad \forall \langle i, j \rangle \in E$$
$$h(g) = 0$$

Proof.

Sketch.

$$h(i) = \kappa v_i = \kappa(v_i - v_j) + \kappa v_j \leq 1 + h(j)$$

$\square$

Corollary

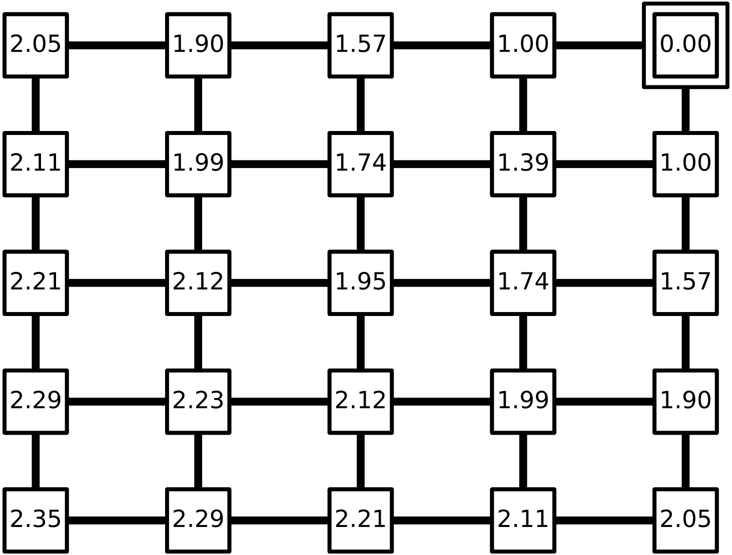
h is a consistent, goal-aware heuristic. [Pommerening et al., 2015]

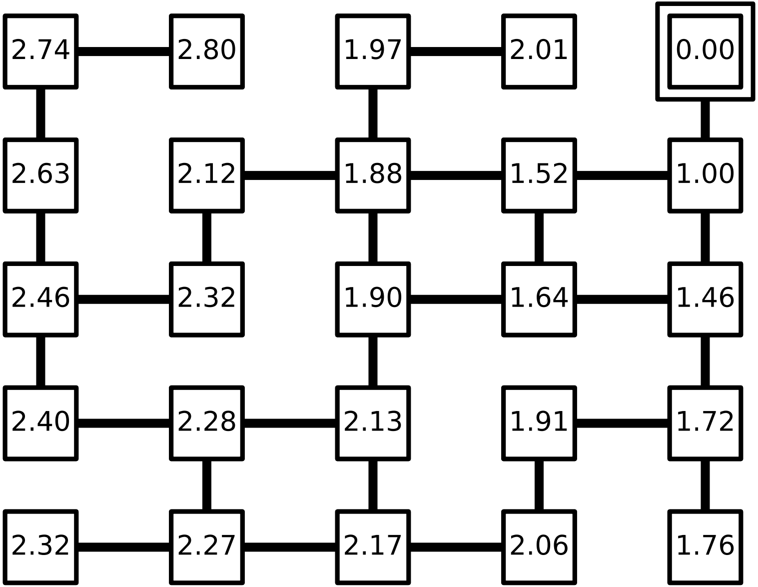
Eigenvector minimises difference of squares

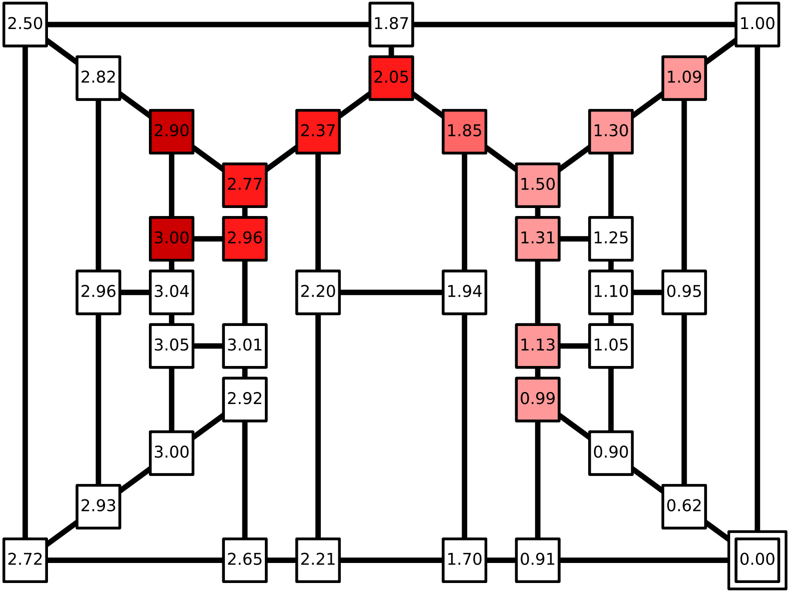
Finding the eigenvector

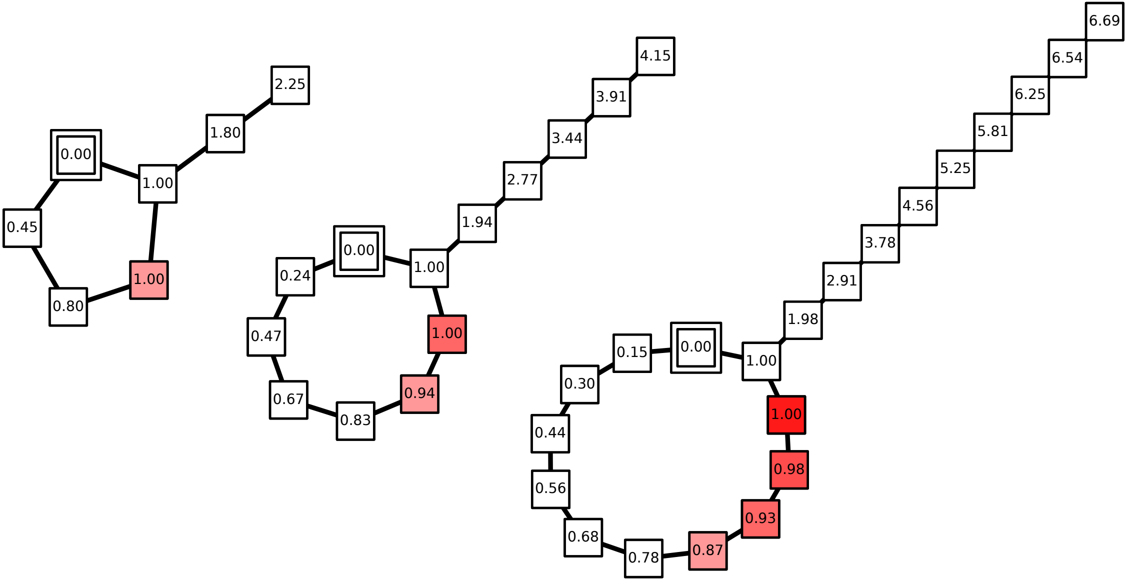
$$\underset{\boldsymbol{x} \neq 0}{\operatorname{argmin}}\, \boldsymbol{x}^T L_g \boldsymbol{x} \text{ s.t. } \|\boldsymbol{x}\| = 1$$

$$\boldsymbol{x}^T L_g \boldsymbol{x} = \sum_{\langle i,j \rangle \in E} (x_i - x_j)^2$$


2.05
1.90
1.57
1.00
0.00
2.11
1.99
1.74
1.39
1.00
2.21
2.12
1.95
1.74
1.57
2.29
2.23
2.12
1.99
1.90
2.35
2.29
2.21
2.11
2.05

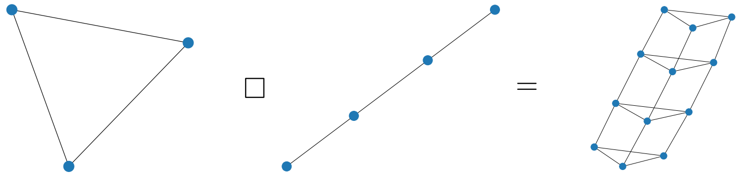

2.74
2.80
1.97
2.01
0.00
2.63
2.12
1.88
1.52
1.00
2.46
2.32
1.90
1.64
1.46
2.40
2.28
2.13
1.91
1.72
2.32
2.27
2.17
2.06
1.76


2.50
1.87
1.00
2.82
2.05
1.09
2.90
2.37
1.85
1.30
2.77
1.50
3.00
2.96
1.31
1.25
2.96
3.04
2.20
1.94
1.10
0.95
3.05
3.01
1.13
1.05
2.92
0.99
3.00
0.90
2.93
0.62
2.72
2.65
2.21
1.70
0.91
0.00


2.25
1.80
0.00
1.00
0.45
1.00
0.80
4.15
3.91
3.44
2.77
1.94
0.00
1.00
0.24
0.47
1.00
0.94
0.67
0.83
6.69
6.54
6.25
5.81
5.25
4.56
3.78
2.91
1.98
0.15
0.00
1.00
0.30
1.00
0.44
0.98
0.56
0.93
0.68
0.78
0.87

Getting transition system's matrix and finding an eigenvector is not realistic...

Cartesian Product for Graphs



Easy to compute eigenvectors for $G \square H$ if we know G and H

Compare to synch. product: $V(G \square H) = V(G) \times V(H) = V(G \otimes H)$

$(i,i') \overset{\square}{\sim} (j,j') \iff (i \overset{G}{\sim} j \wedge i' = j') \vee (i = j \wedge i' \overset{H}{\sim} j')$ $\qquad (i,i') \overset{\otimes}{\sim} (j,j') \iff i \overset{G}{\sim} j \wedge i' \overset{H}{\sim} j'$

$\square$ does not give us much power...

Possible Connections

Finding the eigenvector

$$\operatorname*{argmin}_{\boldsymbol{x}\neq 0} \boldsymbol{x}^T L_g \boldsymbol{x} \text{ s.t. } \|\boldsymbol{x}\| = 1$$

$$\boldsymbol{x}^T L_g \boldsymbol{x} = \sum_{\langle i,j\rangle \in E} (x_i - x_j)^2$$

Potential Heuristics

max objective s.t.

$x_g \leq 0$

$x_i - x_j \leq \text{cost}(i \to j)$

[Pommerening et al., 2015]

Cloth Simulation

Hooke's Law for Elastic Potential

$$U(\delta) = \frac{1}{2} k \delta^2$$

e.g. [Müller et al., 2008]

Reinforcement Learning

Unconstr. Generalised Graph Drawing Objective (1d)

$$\min_{\boldsymbol{u}\in\mathbb{R}} c_i \boldsymbol{u}^T \boldsymbol{L} \boldsymbol{u} \ldots$$

e.g. [Gomez, Bowling, and Machado, 2024]

Summary

1. Can find plans with the spectral method
2. Can think of it as flow (Occupation Measure LP)
3. The eigenvector also induces a consistent, goal-aware heuristic
4. Steinerberger suggests sticking with Dijkstra for now...

What's Next?

- A more powerful graph operation that preserves first eigenvector?
- Explore the connections

`schmlz.github.io`

Gomez, D.; Bowling, M.; and Machado, M. C. 2024. Proper Laplacian Representation Learning. In *Proc. of Int. Conf. on Learning Representations (ICLR)*.

Müller, M.; Stam, J.; James, D.; and Thürey, N. 2008. Real Time Physics Class Notes.

Pommerening, F.; Helmert, M.; Röger, G.; and Seipp, J. 2015. From Non-Negative to General Operator Cost Partitioning. *Proc. of AAAI Conf. on Artificial Intelligence*, 3335–3341.

Puterman, M. L. 2005. *Markov Decision Processes*. Wiley-Interscience.

Seipp, J.; Pommerening, F.; Röger, G.; and Helmert, M. 2016. Correlation complexity of classical planning domains. In *Proc. of Int. Joint Conf. on Artificial Intelligence (IJCAI)*.

Steinerberger, S. 2021. A spectral approach to the shortest path problem. *Linear Algebra and its Applications*, 620: 182–200.